

AUTOMORPHISMS OF INFINITE JOHNSON GRAPH

MARK PANKOV

ABSTRACT. We consider the *infinite Johnson graph* J_∞ whose vertex set consists of all subsets $X \subset \mathbb{N}$ satisfying $|X| = |\mathbb{N} \setminus X| = \infty$ and whose edges are pairs of such subsets X, Y satisfying $|X \setminus Y| = |Y \setminus X| = 1$. An automorphism of J_∞ is said to be *regular* if it is induced by a permutation on \mathbb{N} or it is the composition of the automorphism induced by a permutation on \mathbb{N} and the automorphism $X \rightarrow \mathbb{N} \setminus X$. The graph J_∞ admits non-regular automorphisms. Our first result states that the restriction of every automorphism of J_∞ to any connected component (J_∞ is not connected) coincides with the restriction of a regular automorphism. The second result is a characterization of regular automorphisms of J_∞ as order preserving and order reversing bijective transformations of the vertex set of J_∞ (the vertex set is partially ordered by the inclusion relation). As an application, we describe automorphisms of the associated *infinite Kneser graph*.

1. INTRODUCTION

1.1. Classical Grassmann and Johnson graphs. Let V be an n -dimensional vector space (over a division ring) and $n < \infty$. The *Grassmann graph* $\Gamma_k(V)$ is the graph whose vertex set is the Grassmannian $\mathcal{G}_k(V)$ formed by all k -dimensional subspaces of V and whose edges are pairs of k -dimensional subspaces with $(k-1)$ -dimensional intersections (in what follows, two vertices of a graph joined by an edge will be called *adjacent*). The graph $\Gamma_k(V)$ is connected. By duality, $\Gamma_k(V)$ is isomorphic to $\Gamma_{n-k}(V^*)$ (V^* is the vector space dual to V). Classical Chow's theorem [5] states that every automorphism of $\Gamma_k(V)$, $1 < k < n-1$, is induced by a semilinear automorphism of V or a semilinear isomorphism of V to V^* ; the second possibility can be realized only in the case when $n = 2k$. If $k = 1, n-1$ then any two distinct vertices of $\Gamma_k(V)$ are adjacent and any bijective transformation of the vertex set is an automorphism of $\Gamma_k(V)$. We refer [11] for more information concerning Grassmann graphs.

The *Johnson graph* $J(n, k)$ is formed by all k -element subsets of $\{1, \dots, n\}$, two such subsets are adjacent if their intersection consists of $k-1$ elements. This graph admits a natural isometric embedding in $\Gamma_k(V)$. Consider a base B of the vector space V and the subset of $\mathcal{G}_k(V)$ formed by all k -dimensional subspaces spanned by subsets of B . Subsets of such type are called *apartments* of $\mathcal{G}_k(V)$ (see [11] for motivations of this term). Every apartment of $\mathcal{G}_k(V)$ is the image of an isometric embedding of $J(n, k)$ in $\Gamma_k(V)$. However, the image of every isometric embedding of $J(n, k)$ in $\Gamma_k(V)$ is an apartment of $\mathcal{G}_k(V)$ only in the case when $n = 2k$. This follows from the classification of isometric embeddings of Johnson graphs $J(l, m)$ in $\Gamma_k(V)$ [12]. The graphs $J(n, k)$ and $J(n, n-k)$ are isomorphic: the mapping $*$ transferring every subset $X \subset \{1, \dots, n\}$ to the complement $\{1, \dots, n\} \setminus X$ defines

an isomorphism between these graphs (in the case when $n = 2k$, this is an automorphism of $J(n, k)$). It is not difficult to prove that every automorphism of $J(n, k)$ is induced by a permutation on $\{1, \dots, n\}$ or $n = 2k$ and it is the composition of the automorphism $*$ and the automorphism induced by a permutation on $\{1, \dots, n\}$ (an analog of Chow's theorem).

So, we can say that $J(n, k)$ is a "thin prototype" of $\Gamma_k(V)$. Different characterizations of Grassmann and Johnson graphs can be found in [6, 9, 10], see also Sections 9.1 and 9.3 in [3].

1.2. Grassmann graphs of infinite-dimensional vector spaces. Now, suppose that V is a vector space of infinite dimension \aleph_0 . Grassmannians of V can be defined as the orbits of the action of the linear group $\text{GL}(V)$ on the set of all proper subspaces of V . By [11], there are the following three types of Grassmannians:

- $\mathcal{G}_k(V)$ formed by all subspaces of dimension $k \in \mathbb{N}$,
- $\mathcal{G}^k(V)$ formed by all subspaces of codimension $k \in \mathbb{N}$,
- $\mathcal{G}_\infty(V)$ formed by all subspaces of infinite dimension and codimension.

Let \mathcal{G} be one of these Grassmannians. We say that $S, U \in \mathcal{G}$ are *adjacent* if

$$\dim(S/(S \cap U)) = \dim(U/(S \cap U)) = 1.$$

The associated *Grassmann graph*, it will be denoted by $\Gamma_k(V)$, $\Gamma^k(V)$ or $\Gamma_\infty(V)$ (respectively), is the graph whose vertex set is \mathcal{G} and whose edges are pairs of adjacent elements.

The graph $\Gamma_k(V)$ is connected and every automorphism of $\Gamma_k(V)$ is induced by a semilinear automorphism of V [11]. By duality (see, for example, [1, 11]), $\Gamma^k(V)$ is canonically isomorphic to $\Gamma_k(V^*)$. Thus $\Gamma^k(V)$ is connected and every automorphism of $\Gamma^k(V)$ is induced by a semilinear automorphism of V^* . The graph $\Gamma_\infty(V)$ is not connected. It admits automorphisms whose restrictions to distinct connected components are induced by distinct semilinear isomorphisms [2]. There is the following open problem [8].

Problem. *Describe the restrictions of automorphisms of $\Gamma_\infty(V)$ to connected components.*

The idea used to prove Chow's theorem can not be exploited by many reasons, for example, by the fact that the vector spaces V and V^* are non-isomorphic ($\dim V < \dim V^*$) [1].

1.3. In this paper, a weak version of this problem will be solved. We consider the *infinite Johnson graph* J_∞ — a thin prototype of $\Gamma_\infty(V)$. The vertex set of J_∞ is formed by all subsets $X \subset \mathbb{N}$ satisfying $|X| = |\mathbb{N} \setminus X| = \infty$, two such subsets X, Y are adjacent if

$$|X \setminus Y| = |Y \setminus X| = 1.$$

There is a natural isometric embedding of J_∞ in $\Gamma_\infty(V)$: for any infinite linearly independent subset $B \subset V$ consider the restriction of the graph $\Gamma_\infty(V)$ to the set formed by all elements of $\mathcal{G}_\infty(V)$ spanned by subsets of B .

An automorphism of J_∞ will be called *regular* if it is induced by a permutation on \mathbb{N} or it is the composition of the automorphism induced by a permutation on \mathbb{N} and the automorphism $X \rightarrow \mathbb{N} \setminus X$. The graph J_∞ is not connected and admits non-regular automorphisms (a simple modification of the example from [2]). Our first result (Theorem 1) states that the restriction of every automorphism of J_∞ to

any connected component of J_∞ coincides with the restriction of a regular automorphism. The vertex set of J_∞ is partially ordered by the inclusion relation. The second result (Theorem 2 and Corollary 1) is a characterization of regular automorphisms of J_∞ as order preserving and order reversing bijective transformations of the vertex set. As an application of Theorem 2, we show that every automorphism of the associated *infinite Kneser graph* K_∞ is induced by a permutation on \mathbb{N} .

Some general information concerning automorphisms of graphs can be found in [4].

2. INFINITE JOHNSON GRAPHS

2.1. Definition. Our definition of infinite Johnson graphs is similar to the definition of Grassmann graphs of infinite-dimensional vector spaces given in the previous section.

Denote by S_∞ the group of all permutations on \mathbb{N} and consider the action of this group on the set of all proper subsets of \mathbb{N} . The associated orbits are of the following three types:

- (1) the set consisting of all $X \subset \mathbb{N}$ such that $|X| = k$ (k is a given natural number),
- (2) the set consisting of all $X \subset \mathbb{N}$ such that $|\mathbb{N} \setminus X| = k$ (as in the previous case, k is a given natural number),
- (3) the set consisting of all $X \subset \mathbb{N}$ such that $|X| = |\mathbb{N} \setminus X| = \infty$.

Let J be one of these orbits. We say that $X, Y \in J$ are *adjacent* if

$$|X \setminus Y| = |Y \setminus X| = 1$$

(in the case (1), this condition is equivalent to the equality $|X \cap Y| = k - 1$). The associated *Johnson graph*, we will denote it by J_k , J^k or J_∞ (respectively), is the graph whose vertex set is J and whose edges are pairs of adjacent elements.

2.2. Some remarks on J_k and J^k . The mapping $*$ transferring every subset $X \subset \mathbb{N}$ to the complement $\mathbb{N} \setminus X$ defines an isomorphism between J^k and J_k . The structure of J_k is rather similar to the structure of finite Johnson graphs. This graph is connected. The distance between $X, Y \in J_k$ is equal to $|X \setminus Y| = |Y \setminus X|$ and the diameter of J_k is k . Maximal cliques of J_k are the following two types:

- the *star* $St(A)$, $A \in J_{k-1}$, consisting of all vertices of J_k containing A ,
- the *top* $T(B)$, $B \in J_{k+1}$, consisting of all vertices of J_k contained in B .

Every automorphism f of J_k preserves the class of maximal cliques (stars and tops). Every top consists of precisely $k + 1$ vertices and every star contains an infinite number of vertices; this means that stars go to stars and tops go to tops. In particular, f induces a bijective transformation of the vertex set of J_{k-1} . This transformation is an automorphism of J_{k-1} , since two stars in J_k have a non-zero intersection (consisting of precisely one vertex) if and only if the associated vertices of J_{k-1} are adjacent. So, f induces an automorphism of J_{k-1} . Step by step, we come to a permutation on \mathbb{N} (an automorphism of J_1). This permutation induces f . Now, suppose that g is an automorphism of J^k . Then $h = *g*$ is an automorphism of J_k . Hence h is induced by a permutation $s \in S_\infty$ and an easy verification shows that $g = *h*$ also is induced by s . Therefore, *all automorphisms of the Johnson graphs J_k and J^k are induced by permutations on \mathbb{N} .*

2.3. Basic properties of J_∞ . The graph J_∞ is not connected. The connected component containing $X \in J_\infty$ will be denoted by $J(X)$; it consists of all $Y \in J_\infty$ satisfying

$$|X \setminus Y| = |Y \setminus X| < \infty.$$

Any two connected components of J_∞ are isomorphic (every permutation on \mathbb{N} induces an automorphism of J_∞ , we consider a permutation transferring $X \in J_\infty$ to $Y \in J_\infty$, the associated automorphism of J_∞ sends $J(X)$ to $J(Y)$).

The graph J_∞ contains an infinite number of connected components. If $X \in J_\infty$ and A is a finite subset of X then $X \setminus A$ is a vertex of J_∞ which does not belong to $J(X)$. So, $X, Y \in J_\infty$ belong to distinct connected components if they are incident subsets of \mathbb{N} ($X \subset Y$ or $Y \subset X$).

Let $X \in J_\infty$. The *star* $St(X)$ consists of all $Y \in J_\infty$ containing X and satisfying $|Y \setminus X| = 1$. Similarly, the *top* $T(X)$ is formed by all $Y \in J_\infty$ contained in X and such that $|X \setminus Y| = 1$. Clearly, $St(X)$ and $T(X)$ both are maximal cliques of J_∞ and it is easy to see that *every maximal clique of J_∞ is a star or a top*.

The automorphisms of J_∞ induced by permutations on \mathbb{N} map stars to stars and tops to tops. The automorphism $*$ (sending every $X \in J_\infty$ to $\mathbb{N} \setminus X$) transfers stars to tops and tops to stars.

3. AUTOMORPHISMS OF J_∞

3.1. Main results. Recall that an automorphism of J_∞ is *regular* if it is induced by a permutation on \mathbb{N} or it is the composition of the automorphism $*$ and the automorphism induced by a permutation on \mathbb{N} . Note that $*f = f*$ for every automorphism f of J_∞ induced by a permutation on \mathbb{N} .

The vertex set of J_∞ is partially ordered by the inclusion relation. We say that a bijective transformation f of the vertex set is *order preserving* or *order reversing* if it satisfies the condition

$$X \subset Y \iff f(X) \subset f(Y) \quad \forall X, Y \in J_\infty$$

or the condition

$$X \subset Y \iff f(Y) \subset f(X) \quad \forall X, Y \in J_\infty,$$

respectively. Every automorphism of J_∞ induced by a permutation on \mathbb{N} is order preserving. The automorphism $*$ is order reversing. Therefore, every regular automorphism of J_∞ is order preserving or order reversing; in particular, all regular automorphisms of J_∞ preserve the incidence relation.

Now we modify the example from [2] mentioned above and establish the existence of non-regular automorphisms of J_∞ .

Example 1. Let $A \in J_\infty$ and B be a vertex of the connected component $J(A)$. We take any permutation $s \in S_\infty$ sending A to B . This permutation preserves $J(A)$ and we define

$$f(X) := \begin{cases} s(X) & X \in J(A) \\ X & X \in J_\infty \setminus J(A). \end{cases}$$

This is an automorphism of J_∞ . We choose $Y \in J_\infty$ which is a proper subset of A non-incident with B . It is clear that $Y \notin J(A)$, thus $f(Y) = Y$. This means that f does not preserve the incidence relation (A and Y are incident, but $f(A) = B$ and $f(Y) = Y$ are non-incident). Therefore, the automorphism f is non-regular.

Our main result is the following.

Theorem 1. *The restriction of every automorphism of J_∞ to any connected component of J_∞ coincides with the restriction of a regular automorphism to this connected component.*

The second result is a characterization of regular automorphisms.

Theorem 2. *Every order preserving bijective transformation of the vertex set of J_∞ is the automorphism of J_∞ induced by a permutation on \mathbb{N} .*

Observe that for every order reversing bijective transformation f of the vertex set of J_∞ the mapping $*f$ is order preserving. Thus, as a direct consequence of Theorem 2, we get the following characterization of regular automorphisms of J_∞ .

Corollary 1. *The group of all regular automorphisms of J_∞ coincides with the group formed by all order preserving and order reversing bijective transformations of the vertex set of J_∞ .*

3.2. Application: automorphisms of the infinite Kneser graph. Recall that the *Kneser graph* $K(n, k)$ and the Johnson graph $J(n, k)$ have the same vertex set; two vertices of $K(n, k)$ are adjacent if they are disjoint subsets of $\{1, \dots, n\}$ (here we assume that $k < n - k$). Every automorphism of $K(n, k)$ is induced by a permutation on $\{1, \dots, n\}$. This follows from the Erdős–Ko–Rado theorem; see Section 7.8 in [7].

Consider the *infinite Kneser graph* K_∞ corresponding to the Johnson graph J_∞ . The vertex set of this graph coincides with the vertex set of J_∞ and two vertices of K_∞ are adjacent if they are disjoint subsets of \mathbb{N} . This graph is a thin prototype of so-called *distant* graph defined for a vector space of dimension \aleph_0 [2]. It is not difficult to prove that K_∞ is a connected graph of diameter 3.

Corollary 2. *Every automorphism of K_∞ is induced by a permutation on \mathbb{N} .*

Proof. For every $X \in K_\infty$ denote by X° the set of all vertices of K_∞ adjacent with X . If $X, Y \in K_\infty$ then

$$X \subset Y \iff Y^\circ \subset X^\circ.$$

This implies that every automorphism of K_∞ is an order preserving transformation of the vertex set of K_∞ . Since K_∞ and J_∞ have the same vertex set, Theorem 2 gives the claim. \square

4. PROOF OF THEOREM 1

Let $A \in J_\infty$ and f be the restriction of an automorphism of J_∞ to the connected component $J(A)$. Then $f(J(A))$ is a connected component of J_∞ . It is clear that f transfers maximal cliques of J_∞ (stars and tops) contained in $J(A)$ to maximal cliques contained in $f(J(A))$.

Lemma 1. *One of the following possibilities is realized:*

- (A) *f transfers stars to stars and tops to tops,*
- (B) *f transfers stars to tops and tops to stars.*

Proof. We will use the following facts:

- The intersection of two distinct stars $St(X)$ and $St(Y)$ is empty or contains precisely one vertex; the second possibility is realized only in the case when X, Y are adjacent vertices of J_∞ . The same holds for the intersection of two distinct tops.
- The intersection of a star $St(X)$ and a top $T(Y)$ is empty or consists of precisely two vertices; the second possibility is realized only in the case when $X \subset Y$ and $|Y \setminus X| = 2$.

The proof is a direct verification.

Suppose that $J(A)$ contains a star $St(X)$, $X \in J_\infty$ such that $f(St(X))$ is a star. Consider any $Y \in J_\infty$ adjacent with X . We choose $Z \in J_\infty$ satisfying

$$X \cup Y \subset Z \text{ and } |Z \setminus (X \cup Y)| = 1.$$

Then

$$|St(X) \cap T(Z)| = |St(Y) \cap T(Z)| = 2$$

and

$$|f(St(X)) \cap f(T(Z))| = |f(St(Y)) \cap f(T(Z))| = 2.$$

Since $St(X)$ goes to a star, the latter equality guarantees that $f(T(Z))$ is a top and $f(St(Y))$ is a star.

So, $f(St(Y))$ is a star for every $Y \in J_\infty$ adjacent with X . Now consider an arbitrary $Y \in J_\infty$ such that the star $St(Y)$ is contained in $J(A)$. We take any

$$C_0 \in St(X), \quad C \in St(Y)$$

and consider a path

$$C_0, C_1, \dots, C_i = C$$

in $J(A)$ (a path joining C_0 and C exists, since $J(A)$ is a connected component). Then

$$X, C_0 \cap C_1, C_1 \cap C_2, \dots, C_{i-1} \cap C_i, Y$$

is a path in J_∞ (possibly X coincides with $C_0 \cap C_1$ or $C_{i-1} \cap C_i$ coincides with Y). It was established above that $St(C_0 \cap C_1)$ goes to a star. Then, by the same arguments, the image of $St(C_1 \cap C_2)$ is a star. Step by step, we get that $f(St(Y))$ is a star. Similarly, we establish that tops go to tops.

If f transfers every star to a top then the same arguments show that tops go to stars. \square

Proposition 1. *In the case (A), f is induced by a permutation on \mathbb{N} , i.e. there exists $s \in S_\infty$ such that*

$$f(U) = s(U) \quad \forall U \in J(A).$$

Proposition 1 will be proved in two steps — Lemmas 2 and 3. In each of these lemmas, we assume that f satisfies (A).

For every $X \in J_\infty$ we denote by X^\sim the set consisting of X and all vertices of J_∞ adjacent with X .

Lemma 2. *For every $X \in J(A)$ the restriction of f to X^\sim is induced by a permutation on \mathbb{N} .*

Proof. We can suppose that $f(X) = X$ (otherwise, we consider tf , where $t \in S_\infty$ transfers $f(X)$ to X). In this case, the restriction of f to X^\sim is a bijective transformation of X^\sim .

A star $St(U)$ is contained in X^\sim if and only if

$$(1) \quad U \subset X \text{ and } |X \setminus U| = 1.$$

Thus f defines a permutation on the set of all U satisfying (1). By Subsection 2.2, this permutation is induced by a certain permutation s on X .

Now we extend $s : X \rightarrow X$ to a permutation on \mathbb{N} . Let $n \in \mathbb{N} \setminus X$. We choose $Y \in X^\sim$ containing n . Since $n \notin X$, we have $X \neq Y$ and X, Y are adjacent. This means that n is unique element of $Y \setminus X$. Since $f(Y)$ and $f(X) = X$ are adjacent, $f(Y) \setminus X$ consists of precisely one element. We denote this number by $s(n)$.

Show that our definition of $s(n)$ does not depend on Y . Let us take any $Z \in X^\sim \setminus \{Y\}$ containing n . Since Y and Z both are adjacent to X , we have

$$|X \setminus (X \cap Y)| = |X \setminus (X \cap Z)| = 1.$$

If $X \cap Y$ coincides with $X \cap Z$ then $Y = Z$ (recall that n belongs to both Y, Z and $n \notin X$). Therefore, $X \cap Y$ and $X \cap Z$ are adjacent vertices of J_∞ . The latter guarantees that Y and Z are adjacent. Thus

$$(2) \quad f(Y) = \{s(n)\} \cup (X \cap f(Y)) \text{ and } f(Z) = \{n'\} \cup (X \cap f(Z))$$

are adjacent (here n' is unique element of $f(Z) \setminus X$). Note that

$$(3) \quad X \cap f(Y) \neq X \cap f(Z).$$

Indeed, the equality

$$X \cap f(Y) = X \cap f(Z)$$

implies the existence of a star containing

$$f(X) = X, f(Y), f(Z);$$

however, there is no star containing X, Y, Z (these vertices are contained in a top). Since $f(Y)$ and $f(Z)$ are adjacent, (2) and (3) show that $s(n) = n'$.

It is clear that $s : \mathbb{N} \rightarrow \mathbb{N}$ is a permutation on \mathbb{N} and $f(U) = s(U)$ for every $U \in X^\sim$. \square

So, for every $X \in J(A)$ there is a permutation $s_X \in S_\infty$ such that

$$f(U) = s_X(U) \quad \forall U \in X^\sim.$$

Lemma 3. *If $X, Y \in J(A)$ are adjacent then $s_X = s_Y$.*

Proof. Suppose that

$$X = \{n\} \cup (X \cap Y) \text{ and } Y = \{m\} \cup (X \cap Y).$$

We can assume that $f(X) = X$ and $f(Y) = Y$. Indeed, in the general case we have

$$f(X) = \{n'\} \cup (f(X) \cap f(Y)), \quad f(Y) = \{m'\} \cup (f(X) \cap f(Y))$$

and consider tf , where $t \in S_\infty$ transfers n', m' and $f(X) \cap f(Y)$ to n, m and $X \cap Y$, respectively.

It is easy to see that

$$X^\sim \cap Y^\sim = St(X \cap Y) \cup T(X \cup Y).$$

We have

$$s_X(X \cap Y) = s_X(X) \cap s_X(Y) = f(X) \cap f(Y) = X \cap Y.$$

Similarly, we get

$$s_Y(X \cap Y) = X \cap Y.$$

Then $s_X(n) = s_Y(n) = n$ and $s_X(m) = s_Y(m) = m$.

Let $k \in \mathbb{N} \setminus (X \cap Y)$. Then

$$U := \{k\} \cup (X \cap Y) \in St(X \cap Y) \subset X^\sim \cap Y^\sim$$

and

$$s_X(U) = \{s_X(k)\} \cup (X \cap Y), \quad s_Y(U) = \{s_Y(k)\} \cup (X \cap Y).$$

The equality

$$s_X(U) = f(U) = s_Y(U)$$

shows that $s_X(k) = s_Y(k)$.

Let $k \in X \cap Y$. Then

$$W := \{n, m\} \cup [(X \cap Y) \setminus \{k\}] \in T(X \cup Y) \subset X^\sim \cap Y^\sim$$

and

$$s_X(W) = \{n, m\} \cup [(X \cap Y) \setminus \{s_X(k)\}], \quad s_Y(W) = \{n, m\} \cup [(X \cap Y) \setminus \{s_Y(k)\}].$$

The equality

$$s_X(W) = f(W) = s_Y(W)$$

implies that $s_X(k) = s_Y(k)$. \square

By connectedness, Lemma 3 guarantees that $s_X = s_Y$ for all $X, Y \in J(A)$. Proposition 1 is proved.

In the case (B), the mapping $*f$ transfers stars to stars and tops to tops; hence it is induced by a permutation on \mathbb{N} . Thus f is the composition of $*$ and the mapping induced by a permutation on \mathbb{N} .

5. PROOF OF THEOREM 2

Let f be an order preserving bijective transformation of the vertex set of J_∞ .

Lemma 4. *Let $\{X_i\}_{i \in I}$ be a family of vertices of J_∞ (possibly infinite) such that*

$$X := \bigcap_{i \in I} X_i \in J_\infty$$

Then

$$f(X) = \bigcap_{i \in I} f(X_i).$$

Proof. Since f is order preserving, $f(X)$ is contained in every $f(X_i)$ and we have

$$(4) \quad f(X) \subset \bigcap_{i \in I} f(X_i).$$

The inclusion

$$f(X) \subset \bigcap_{i \in I} f(X_i) \subset f(X_i)$$

and the fact that $f(X), f(X_i)$ are vertices of J_∞ guarantee that

$$X' := \bigcap_{i \in I} f(X_i) \in J_\infty.$$

The inverse mapping f^{-1} is order preserving and $f^{-1}(X')$ is contained in every X_i . Thus

$$f^{-1}(X') \subset X \quad \text{and} \quad X' \subset f(X).$$

By (4), $f(X) \subset X'$. Therefore, $f(X) = X'$. \square

Lemma 5. *If $X, Y \in J_\infty$, $Y \subset X$ and $|X \setminus Y| = 1$ then*

$$|f(X) \setminus f(Y)| = 1.$$

Proof. It is clear that $f(Y)$ is a proper subset of $f(X)$. If $|f(X) \setminus f(Y)| > 1$ then there exists $Z \in J_\infty \setminus \{f(X), f(Y)\}$ such that

$$f(Y) \subset Z \subset f(X).$$

Then

$$Y \subset f^{-1}(Z) \subset X.$$

Since $|X \setminus Y| = 1$, the latter inclusions mean that $f^{-1}(Z)$ coincides with X or Y , a contradiction. \square

Lemma 6. *For every $X \in J_\infty$ there exists a bijective mapping $s : X \rightarrow f(X)$ such that $f(Y) = s(Y)$ for every $Y \in J_\infty$ contained in X .*

Proof. We restrict ourself to the case then $f(X) = X$ (in the general case, we consider the mapping tf with $t \in S_\infty$ transferring $f(X)$ to X). Denote by \mathcal{X} the set of all $Y \subset X$ satisfying $|X \setminus Y| = 1$. All elements of \mathcal{X} are vertices of J_∞ and, by Lemma 5, f defines a permutation on \mathcal{X} . By Subsection 2.2, this permutation is induced by a certain permutation s on X , i.e.

$$f(Y) = s(Y) \quad \forall Y \in \mathcal{X}.$$

Every $Y \in J_\infty$ contained in X can be presented as the intersection of a family $\{Y_i\}_{i \in I}$ of elements from \mathcal{X} (possible infinite). Then

$$f(Y) = \bigcap_{i \in I} f(Y_i) = \bigcap_{i \in I} s(Y_i) = s(Y)$$

(the first equality follows from Lemma 4). \square

For every $n \in \mathbb{N}$ denote by $[n]$ the set of all vertices of J_∞ containing n .

Lemma 7. *For every $n \in \mathbb{N}$ there exists $s(n) \in \mathbb{N}$ such that*

$$f([n]) = [s(n)].$$

Proof. We take $Y_1, Y_2 \in J_\infty$ satisfying

$$Y_1 \cup Y_2 \in J_\infty \quad \text{and} \quad Y_1 \cap Y_2 = \{n\}.$$

Let $X := Y_1 \cup Y_2$. Lemma 6 implies the existence of a bijection $s : X \rightarrow f(X)$ such that $f(Y) = s(Y)$ for every $Y \in J_\infty$ contained in X . Then

$$f(Y_1) \cap f(Y_2) = s(Y_1) \cap s(Y_2) = s(Y_1 \cap Y_2) = \{s(n)\}.$$

We show that the number $s(n)$ is as required.

Let $Z \in [n]$. If Z has an infinite intersection with X then $Z \cap X$ is an element of $[n]$ contained in X and

$$f(Z \cap X) = s(Z \cap X)$$

contains $s(n)$. The inclusion

$$f(Z \cap X) \subset f(Z)$$

guarantees that $f(Z) \in [s(n)]$.

Suppose that $Z \cap X$ is finite. In this case, we decompose $Z \setminus X$ in the disjoint union of two infinite subsets A, B and define

$$T := (Z \cap X) \cup A.$$

Then $T \in J_\infty$; moreover,

$$n \in T \subset Z \text{ and } X' := X \cup T \in J_\infty.$$

By Lemma 6, there exists a bijection $s' : X' \rightarrow f(X')$ such that $f(Y) = s'(Y)$ for every $Y \in J_\infty$ contained in X' . Since

$$Y_1 \cap Y_2 \cap T \neq \emptyset$$

(this intersection contains n), we have

$$f(Y_1) \cap f(Y_2) \cap f(T) = s'(Y_1) \cap s'(Y_2) \cap s'(T) = s'(Y_1 \cap Y_2 \cap T) \neq \emptyset.$$

On the other hand,

$$f(Y_1) \cap f(Y_2) = \{s(n)\}.$$

Hence $f(T)$ contains $s(n)$ and the inclusion $f(T) \subset f(Z)$ implies that $f(Z)$ belongs to $[s(n)]$.

So, we obtain that $f([n]) \subset [s(n)]$. Applying the same arguments to the transformation f^{-1} , we get the inverse inclusion. \square

The mapping $n \rightarrow s(n)$ is a permutation on \mathbb{N} and f is the automorphism of J_∞ induced by this permutation.

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DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF WARMIA AND MAZURY,
ŻOLNIERSKA 14A, 10-561 OLSZTYN, POLAND

E-mail address: markpankov@gmail.com, pankov@matman.uwm.edu.pl